STRONG LEFSCHETZ PROPERTY UNDER REDUCTION

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Abstract. Let $n > 1$ and G be the group $\mathrm{SU}(n)$ or $\mathrm{Sp}(n)$. This paper constructs compact symplectic manifolds whose symplectic quotient under a Hamiltonian G-action does not inherit the strong Lefschetz property.

1. Introduction

Let (M, Ω) be a compact symplectic manifold of dimension $2m$. For each $0 \leq k \leq m$, the symplectic form Ω induces a map

$$\Omega^k \colon \mathrm{H}^{m-k}(M) \to \mathrm{H}^{m+k}(M), \qquad [\alpha] \mapsto [\Omega^k \wedge \alpha].$$

The manifold (M, Ω) is called Lefschetz if Ω^{m-1} is an isomorphism. It is called strong Lefschetz if Ω^k is an isomorphism for all $0 \leq k \leq m$. By Poincaré Duality, the Lefschetz maps are surjective if and only if they are injective.

The simplest examples of symplectic manifolds are Kähler manifolds. By the Strong Lefschetz Theorem, all compact Kähler manifolds are strong Lefschetz. On the other hand, Thurston and others constructed compact symplectic manifolds that do not admit Kähler structures [1, 5, 6, 16, 20], either by showing that these manifolds have an odd Betti number which is odd, or that they do not have the strong Lefschetz property.

Another interesting fact about the Lefschetz property concerns Hamiltonian actions. A Theorem of Ono [18] asserts that if a symplectic manifold (M, Ω) is Lefschetz, then a symplectic circle action on (M, Ω) is Hamiltonian if and only if it has fixed points.

We recall that an action of a compact Lie group G on (M, Ω) is symplectic if it preserves the symplectic form Ω. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{g}^*$ its dual. For any $\xi \in \mathfrak{g}$, denote by ξ_M the vector field on M induced by the group action of $\exp(t\xi)$. A symplectic action is Hamiltonian if there exists a moment map $\Phi \colon M \to \mathfrak{g}^*$, which is equivariant with respect to the group action on M and the coadjoint action on $\mathfrak{g}^*$ such that

$$d\langle \Phi, \xi \rangle = \iota(\xi_M)\Omega$$

for all $\xi \in \mathfrak{g}$. The orbit space $\Phi^{-1}(0)/G$ is called the symplectic quotient, and often denoted by $M /\!\!/ G$. If G acts freely on $\Phi^{-1}(0)$, the symplectic quotient $M /\!\!/ G$ is a smooth symplectic manifold. Its symplectic form ω is defined by

$$\pi^* \omega = i^* \Omega,$$

where $\pi\colon \Phi^{-1}(0) \to M/\!\!/G$ is the orbit map, and $i\colon \Phi^{-1}(0) \hookrightarrow M$ is the inclusion. In general, the symplectic quotient is a symplectic stratified space [13]. Suppose that the original manifold (M, Ω) is Kähler, and that the action preserves both the Kähler metric and the symplectic form, hence, the complex structure. Then the symplectic quotient $M/\!\!/G$ is Kähler with Kähler form ω [9, 11].

Products of compact non-Lefschetz symplectic manifolds with a two-sphere rotated about a fixed axis are trivial examples such that the existence of a non-Lefschetz symplectic quotient implies the original manifold is not Lefschetz. However, Lin [14] constructed examples of Hamiltonian S^1-manifolds (M, Ω) that are strong Lefschetz but their symplectic quotients $(M/\!\!/S^1, \omega)$ are not. In this paper, we generalize Lin's result.

Let $\omega_{\mathbb{C}^n} = \frac{i}{2}\sum dz_j \wedge d\bar{z}_j = \sum dx_j \wedge dy_j$ be the standard symplectic form on $\mathbb{C}^n$. Let G be a compact Lie group with an n-dimensional unitary representation $\tau\colon G \to \mathrm{U}(n)$. Then $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ is a Hamiltonian G-manifold.

Theorem 1.1. *Suppose $\mathbb{C}^n/\!\!/G$ consists of a single point. Then there exists a compact connected Hamiltonian G-manifold (M, Ω) that is strong Lefschetz, but its symplectic quotient $(M/\!\!/G, \omega)$ is not.*

As an application, let $m, k = 2, 3, 4, \ldots$ and G be either $\mathrm{SU}(m)$ or $\mathrm{Sp}(k)$. Then the moment map for the standard G-representation on $\mathbb{C}^n$, for $n = m$ and $n = 2k$, respectively, is

$$\psi\colon \mathbb{C}^n \to \mathfrak{g} \cong \mathfrak{g}^* \qquad \text{given by} \qquad \psi(z) = \frac{i}{2}zz^*,$$

where we identify the Lie algebra $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ via the inner product $\langle A, B\rangle = \mathrm{trace}(A^*B)$. A direct calculation shows that $\psi^{-1}(0) = \{0\}$. Theorem 1.1 then implies

Theorem 1.2. *Let $n > 1$ and let G be either $\mathrm{SU}(n)$ or $\mathrm{Sp}(n)$. Then there exists a compact connected Hamiltonian G-manifold (M, Ω) that is strong Lefschetz, but its symplectic quotient $(M/\!\!/G, \omega)$ is not.*

In Section 2, we spell out the construction for the manifold (M, Ω) in Theorem 1.1. In Section 3, we compute the Lefschetz maps. In our proof, we use a result of Gompf [6] as our symplectic quotient:

Theorem 1.3 (Gompf)**.** *There exists a 4-dimensional symplectic manifold (B, ω) such that the Lefschetz map $\omega\colon \mathrm{H}^1(B) \to \mathrm{H}^3(B)$ is the zero-map. Moreover, there exists an integral class $c \in \mathrm{H}^2(B)$ such that $c\colon \mathrm{H}^1(B) \to \mathrm{H}^3(B)$ is an isomorphism.*

The strong Lefschetz property also has an equivalent description in terms of symplectic harmonic forms. For details, see [3, 15, 21]. The Hamiltonian G-manifolds (M, Ω) we construct here have the property that each de Rham cohomology class contains a symplectic harmonic representative while the same is not true for $(M/\!\!/G, \omega)$.

2. Construction

Let G be a compact Lie group acting on $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ by a unitary representation with the property that the symplectic quotient $\mathbb{C}^n /\!\!/ G$ is a point. Extending the action to $\mathbb{C}^n \times \mathbb{C}$ trivially on the last factor, it induces a Hamiltonian G-action on the complex projective space $\mathbb{P}(\mathbb{C}^n \times \mathbb{C}) = \mathbb{CP}^n$. Let $F = \mathbb{CP}^n$ and ω_F the Fubini-Study symplectic form on F scaled by $\epsilon > 0$. The symplectic quotient $F /\!\!/ G$ under the induced G-action is again a point.

Remark 2.1. This projective space (F, ω_F) can be constructed by symplectic reduction. Let S^1 denote the circle group and we identify both the Lie algebra of S^1 and its dual with $\mathbb{R}$. The multiplication of S^1 on $\mathbb{C}^n \times \mathbb{C}$ is a Hamiltonian action with moment map $\phi\colon \mathbb{C}^n \times \mathbb{C} \to \mathbb{R}$ given by

$$\phi(z, w) = \frac{1}{2}(-\|z\|^2 - |w|^2 + \text{constant}).$$

If we choose the constant to be ϵ, the symplectic quotient is (F, ω_F). This is the same as taking a closed ball in $\mathbb{C}^n$ of radius $\epsilon^{1/2}$ centered at zero and reduce the boundary since

$$\begin{aligned} \phi^{-1}(0)/S^1 &= \left\{ \|z\|^2 + |w|^2 = \epsilon \right\} / S^1 \\ &\cong \left\{ \|z\|^2 = \epsilon \right\} / S^1 \sqcup \left\{ \|z\|^2 < \epsilon \right\}. \end{aligned}$$

Let (B, ω) be a symplectic manifold and let $\pi\colon P \to B$ be a principal S^1-bundle over B with a given Chern class c and θ a connection form on P. With the Hamiltonian S^1-action on (F, ω_F) induced from the multiplication on $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$, we form the associated bundle $P \times_{S^1} F$.

Lemma 2.2. *For ϵ sufficiently small, the associated bundle $P \times_{S^1} F$ is a symplectic manifold with fiber (F, ω_F).*

Proof. This is done by the technique of minimal coupling [19, 23].

First identify both the Lie algebra of S^1 and its dual with $\mathbb{R}$ and consider the space $P \times \mathbb{R}$. The minimal coupling form

$$\pi^*\omega + d\langle \mathrm{pr}_2, \theta \rangle$$

is nondegenerate on $P \times \{0\}$, and therefore is symplectic on a small δ-neighborhood $P \times (-\delta, \delta)$. Let $I = (-\delta, \delta)$. The S^1-action on $P \times I$ given by $a \cdot (p, \eta) = (pa^{-1}, \mathrm{Ad}^*(a)\eta)$ is Hamiltonian with minus the projection onto the second factor

$$-\mathrm{pr}_2\colon P \times I \to \mathbb{R}$$

being its moment map. The symplectic quotient $(P \times I) /\!\!/ S^1$ is (B, ω).

Second consider the product space $P \times I \times F$. Let $\Phi_F\colon F \to \mathbb{R}$ be the S^1-moment map on F. The diagonal S^1-action on $(P \times I) \times F$ is Hamiltonian with moment map given by

$$\Phi(p, \eta, f) = \Phi_F(f) - \eta.$$

If $\epsilon < \delta$, the image of F under Φ_F is contained in I, and $\Phi^{-1}(0) \cong P \times F$. So the symplectic quotient is

$$\Phi^{-1}(0)/S^1 = P \times_{S^1} F$$

with the symplectic form

$$\Omega([p, f]) = \pi^* \omega(p) + \langle \Phi_F(f), \Theta(p) \rangle + \omega_F(f),$$

where $\Theta = d\theta - \frac{1}{2}[\theta, \theta]$ is the curvature form of θ. Since θ is a connection 1-form on the principal S^1-bundle P with Chern class c, it follows that $\pi^* c = -[\Theta]$.

□

Finally, since the G-action on $\mathbb{C}^n$ we start out with commutes with the S^1-multiplication, the induced Hamiltonian actions on F also commute. The G-moment map on F is S^1-invariant. Hence the manifold $(P \times_{S^1} F, \Omega)$ inherits a fiberwise G-action with moment map

$$\Psi \colon P \times_{S^1} F \to \mathfrak{g}^*$$

induced by the G-moment map on the fiber. Reduction can be carried out fiber by fiber. Since $F /\!\!/ G$ is a point, by construction, $\Psi^{-1}(0)/G = (B, \omega)$.

Lemma 2.3. *Let $M = P \times_{S^1} F$ and Ω be as in the proof of Lemma 2.2. Then (M, Ω) is a Hamiltonian G-manifold with $M /\!\!/ G = (B, \omega)$.*

For our purpose, we use the symplectic manifold (B, ω) and the Chern class c prescribed in Theorem 1.3.

3. Strong Lefschetz property

This section proves that the Hamiltonian G-manifold (M, Ω) constructed in the previous section is strong Lefschetz. Namely, we show that the map

$$\Omega^k \colon \mathrm{H}^{n+2-k}(M) \to \mathrm{H}^{n+2+k}(M)$$

is an isomorphism for every integer $0 \le k \le n+2$. By Poincaré Duality, it suffices to show that the map Ω^k is injective. This holds for $k = n+2$ by the nondegeneracy of the symplectic form and also for $k = 0$. It remains to show injectivity for $1 \le k \le n+1$.

A typical fibre of (M, Ω) is (F, ω_F), which is isomorphic to $\mathbb{CP}^n$ with a scaled Fubini-Study symplectic form. To simplify notations, let

$$x = \pi^* \omega, \quad y = \langle \Phi_F, \Theta \rangle, \quad z = \omega_F, \quad u = \langle \Phi_F, \Theta \rangle + \omega_F = y + z.$$

When there is no confusion, we use the same symbol to denote a closed form and its cohomology class. For example, x and u are both closed and we use the same symbol for both the form and its cohomology class in the subsequent computation.

In this notation, $\mathrm{H}^*(F)$ is generated by the class $u|_F \in \mathrm{H}^2(F)$. Hence $\mathrm{H}^*(M)$ as a $\mathrm{H}^*(B)$-module is freely generated by $\{1, u, u^2, \ldots, u^n\}$ by the Leray-Hirsch Theorem (see, for example, §5 of [2]):

Theorem 3.1 (Leray-Hirsch). *Let M be a fiber bundle over B with fiber F. The cohomology $\mathrm{H}^*(M)$ is a free $\mathrm{H}^*(B)$-module generated by $\{e_1, \cdots, e_r\} \subset \mathrm{H}^*(M)$ if the restriction of $\{e_1, \cdots, e_r\}$ to F generates $\mathrm{H}^*(F)$.*

For the rest of the paper, we identify the elements in $\mathrm{H}^*(B)$ and $\mathrm{H}^*(F)$ with their images in $\mathrm{H}^*(M) \cong \mathrm{H}^*(B) \otimes \mathrm{H}^*(F)$. By Theorem 3.1, a class $\alpha \in \mathrm{H}^k(M)$ can be written as

$$\alpha = \begin{cases} b_4 u^{(k-4)/2} + b_2 u^{(k-2)/2} + b_0 u^{k/2}, & \text{if } k \text{ is even,} \\ b_3 u^{(k-3)/2} + b_1 u^{(k-1)/2}, & \text{if } k \text{ is odd,} \end{cases}$$

where b_j denotes both a class in $\mathrm{H}^j(B)$ and its lifting in $\mathrm{H}^j(M)$.

In particular,

$$u^{n+1} = \beta_4 u^{n-1} + \beta_2 u^n = \frac{n(n+1)}{2} y^2 z^{n-1} + (n+1) y z^n$$

for some $\beta_2 \in \mathrm{H}^2(B)$ and $\beta_4 \in \mathrm{H}^4(B)$.

Let $h \in \mathrm{H}^2(B)$. Then

$$\begin{aligned} u^{n+1} h &= (\beta_4 u^{n-1} + \beta_2 u^n) h = \beta_2 h u^n = \beta_2 h z^n \\ &= \left(\frac{n(n+1)}{2} y^2 z^{n-1} + (n+1) y z^n \right) h = (n+1) y h z^n. \end{aligned}$$

The form z^n restricted to each fibre of the bundle projection $M \to B$ is the volume form on (F, ω_F). Integrating along the fibre, we obtain

$$\int_M \beta_2 h z^n = V(\omega_F) \int_B \beta_2 h,$$

where $V(\omega_F) = \frac{\pi^n \epsilon^n}{\Gamma(n+1)}$ is the volume and Γ is the Euler Γ-function. On the other hand,

$$\int_M (n+1) y h z^n = \int_M (n+1) \langle \Phi_F, \Theta \rangle h z^n = \int_M -(n+1) \langle \Phi_F, \pi^* c \rangle h z^n.$$

In this case, along each fibre, the volume form ω_F^n is scaled by the moment map Φ_F. The volume of S^{2n-1} with radius r is $\frac{2\pi^n r^{2n-1}}{\Gamma(n)}$. By Remark 2.1, it follows that

$$\begin{aligned} \int_M (n+1) y h z^n &= \left[\int_0^{\epsilon^{1/2}} \frac{r^2}{2} \cdot \frac{2\pi^n r^{2n-1}}{\Gamma(n)} dr \right] \int_B (n+1) c h \\ &= \frac{\pi^n \epsilon^{n+1}}{(2n+2)\,\Gamma(n)} \int_B (n+1) c h \\ &= \frac{\pi^n \epsilon^{n+1}}{2\,\Gamma(n)} \int_B c h\,. \end{aligned}$$

Hence $\beta_2 = \frac{1}{2} n \epsilon c$. Note that β_2 depends on ϵ which can be chosen anywhere between 0 and δ.

Similarly,

$$u^{n+2} = (\beta_4 + \beta_2^2)u^n = \frac{(n+1)(n+2)}{2} y^2 z^n.$$

Integrating, we get

$$\begin{aligned}\int_M u^{n+2} &= \int_M (\beta_4 + \beta_2^2)u^n = \int_M (\beta_4 + \beta_2^2) z^n = V(\omega_F)\int_B (\beta_4 + \beta_2^2)\\ &= \int_M \frac{(n+1)(n+2)}{2} y^2 z^n = \frac{(n+1)\pi^n \epsilon^{n+2}}{8\,\Gamma(n)} \int_B c^2.\end{aligned}$$

Hence $\beta_4 = \frac{1}{8} n(1-n)\epsilon^2 c^2$.

Now we are ready to compute the kernel of the Lefschetz map

$$\Omega^k \colon \mathrm{H}^{n+2-k}(M) \to \mathrm{H}^{n+2+k}(M).$$

Let $\alpha \in \mathrm{H}^{n+2-k}(M)$. We will show that $\Omega^k \alpha = 0$ implies that $\alpha = 0$.

With the notations explained earlier, we have

$$\begin{aligned}\Omega^k &= (x+u)^k\\ &= \begin{cases} x+u, & \text{if } k=1,\\ \frac{k(k-1)}{2} x^2 u^{k-2} + kxu^{k-1} + u^k, & \text{if } 2 \le k \le n,\\ \left(\frac{n(n-1)}{2} x^2 + \beta_4\right) u^{n-1} + ((n+1)x + \beta_2)u^n, & \text{if } k = n+1.\end{cases}\end{aligned}$$

If $k = n+1$, we have $\alpha = b_1$. Hence,

$$\Omega^{n+1}\alpha = ((n+1)x + \beta_2) b_1 u^n.$$

If α is in the kernel, that is, $\Omega^{n+1}\alpha = 0$, then

$$((n+1)x + \beta_2) b_1 = 0.$$

Since $(n+1)x + \beta_2 = (n+1)x + \frac{1}{2} n\epsilon c$ defines an injective map from $\mathrm{H}^1(B)$ to $\mathrm{H}^3(B)$ for a generic ϵ, we conclude that $b_1 = 0$. And therefore $\ker(\Omega^{n+1}) = \{0\}$ for a generic ϵ.

If $k = n$, we have $\alpha = b_2 + b_0 u$. Hence,

$$\begin{aligned}\Omega^n \alpha &= \left(b_2 nx + b_0 \frac{n(n-1)}{2} x^2\right) u^{n-1} + (b_2 + b_0 nx)u^n + b_0 u^{n+1}\\ &= \left(b_2 nx + b_0 \frac{n(n-1)}{2} x^2 + b_0\beta_4\right) u^{n-1} + (b_2 + b_0 nx + b_0 \beta_2)u^n.\end{aligned}$$

If α is in the kernel, that is, $\Omega^n\alpha = 0$, then

$$\begin{cases} b_2 nx + b_0(\frac{n(n-1)}{2} x^2 + \beta_4) &= 0,\\ b_2 + b_0(nx + \beta_2) &= 0.\end{cases}$$

Assume that $b_0 \neq 0$. Since $b_2 = -b_0(nx+\beta_2)$, we get

$$\begin{aligned} 0 &= -b_0(nx+\beta_2)nx + b_0\left(\frac{n(n-1)}{2}x^2+\beta_4\right) \\ &= \left(\beta_4 - n\beta_2 x - \frac{n(n+1)}{2}x^2\right)b_0. \end{aligned}$$

This implies that

$$\frac{n(1-n)\epsilon^2}{8}c^2 - \frac{n^2\epsilon}{2}cx - \frac{n(n+1)}{2}x^2 = 0.$$

However the left hand side is lifted from an element in $\mathrm{H}^4(B)$ and can be made non-zero with a generic ϵ. This is a contradiction. Hence for a generic ϵ, we must have $b_0 = 0$ and $b_2 = -b_0(\beta_2+nx) = 0$. We conclude that $\ker(\Omega^n) = \{0\}$ for a generic ϵ.

If $1 \leq k < n$ and $n-k$ is even, we have

$$\alpha = b_4 u^{(n-k-2)/2} + b_2 u^{(n-k)/2} + b_0 u^{(n-k+2)/2}.$$

Hence,

$$\begin{aligned} \Omega^k\alpha = &\left(b_4 + b_2 kx + b_0\frac{k(k-1)}{2}x^2\right)u^{(n+k-2)/2} \\ &+ (b_2+b_0kx)u^{(n+k)/2} + b_0u^{(n+k+2)/2}. \end{aligned}$$

Suppose $\Omega^k\alpha = 0$. Then

$$\begin{cases} b_4 + b_2kx + b_0\frac{k(k-1)}{2}x^2 &= 0, \\ b_2 + b_0kx &= 0, \\ b_0 &= 0. \end{cases}$$

This implies that $b_0 = b_2 = b_4 = 0$. Hence $\ker(\Omega^k) = \{0\}$.

If $1 \leq k < n$ and $n-k$ is odd, we have

$$\alpha = b_3 u^{(n-k-1)/2} + b_1 u^{(n-k+1)/2}.$$

Hence,

$$\Omega^k\alpha = (b_3 + b_1kx)u^{(n+k-1)/2} + b_1u^{(n+k+1)/2}.$$

Suppose $\Omega^k\alpha = 0$. Then

$$\begin{cases} b_3 + b_1kx &= 0, \\ b_1 &= 0. \end{cases}$$

This implies that $b_1 = b_3 = 0$. Hence $\ker(\Omega^k) = \{0\}$.

Hence we conclude that for a generic ϵ, the Lefschetz map Ω^k is an isomorphism for all

$$1 \leq k \leq n+1.$$

This completes the proof of Theorem 1.1. Our manifold (M,Ω) is strong Lefschetz while its symplectic quotient (B,ω) is not.

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